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ARE THERE FREE GROUPS IN DIVISION RINGS?

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ABSTRACT

The following conjecture is investigated: a noncentral subnormal subgroup of the multiplicative group of a division ring contains a noncyclic free subgroup. Special cases are proved, entailing several known commutativity theorems. Also a new framework is presented for some kinds of commutativity theorems, based on the existence of (group) words for which one can always find an appropriate substitution by elements of such a subnormal subgroup that yields a noncentral element. Several families of such words are given; one gets commutativity theorems imposing some restrictions (like periodicity) to the image of these words.

1. Introduction

We are concerned with a problem posed by Lichtman [8], which, at least for discussion purposes, we formulate as:

CONJECTURE 1. *The multiplicative group of a noncommutative division ring contains a noncyclic free subgroup.*

A stronger conjecture will actually be approached:

CONJECTURE 2. *Any noncentral subnormal subgroup of the multiplicative group of a division ring contains a noncyclic free subgroup.*

An affirmative answer to either of these would be an umbrella for several known theorems. For instance, in [4], Chapter 3, there are several group-theoretic conditions whose occurrence in the multiplicative group of a division ring entails commutativity of the ring; those conditions, to which we return later, are impossible in the presence of a noncommutative free subgroup, hence

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commutativity would follow from Conjecture 1. (There are other commutativity theorems based on algebraic conditions, however, which are apparently unrelated to this conjecture.) Further work of Herstein [5, 6], Stuth [12] and others extend some of these results to conditions on a subnormal subgroup which imply it is central; these theorems would easily follow from Conjecture 2 (actually, these papers are probably stepping stones in any attempt to settle the Conjecture). As direct evidence towards these conjectures, we quote Gonçalves [2], Lichtman [9, 10], and the present work.

On this line, the main results appear in Section 3, and they can be put together as:

THEOREM. *Conjecture* 2 *holds for a subnormal subgroup which contains a noncentral element* x *such that either* (a) x *is algebraic over the center* Z *of the ring, and the field* $Z(x)$ *admits a nontrivial* Z*-automorphism, or* (b) $x^p \in Z$, *where* $p = 2$ *or* $p = \operatorname{char} Z$ (>0).

The conjectures suggest a new perspective on some commutativity theorems. Let w be a word of the free group on n generators. For any group G, w induces, by substitution, a map of the n-fold cartesian product $G \times \cdots \times G$ to G, which for short we call the *word map* w on G. Say that w is *noncentral* for G if its image is not contained in the center $C(G)$; equivalently, if the relation $w = 1$ does not hold in $G/C(G)$. Clearly, if G contains a nonabelian free subgroup, every word is noncentral for G. Say that w is an N-word (SN-word) provided it is noncentral for every G that is (a noncentral subnormal subgroup of) the multiplicative group of a noncommutative division ring. Sections 4 and 5 present several such words (actually, all words should be SN). Many of the aforementioned commutativity theorems make statements about the image of an SN word map; they are covered in Section 4.

We fix some notation throughout: D is a division ring with center Z, their multiplicative subgroups are D^*, Z^*; N is a subnormal subgroup of D^*. We abbreviate "contains a noncyclic free group" to "contains a free group". The group theoretic commutator $xyx^{-1}y^{-1}$ will be denoted (x, y), while $[x, y] = xy - yx$. We also use the iterated inner derivation $[x, y]^{(k)}$ given by: $[x, y]^{(0)} = x$, $[x, y]^{(k)} = [[x, y]^{(k-1)}, y]$.

2. A computational tool kit

The combined use of the lemmas at this section is at the core of most results in this paper. The first one is a takeoff from Herstein [6]. We denote by $C_X(p)$ the coefficient of X in p, where p is a polynomial in the indeterminate X.

DESCENDING LEMMA. *Let $a, u \in D^*$, and suppose that $u^n = \alpha \in Z$ for some $n > 0$. Consider the sequences of polynomials $p_i = p_i(a, u; X)$, $\bar{p}_i \in D[X]$, $q_i \in Z[X^n]$, $i = 0, 1, \ldots$ defined by:*

$$p_0 = (1 - uX)(1 - \alpha X^n), \quad \bar{p}_0 = 1 + uX + u^2X^2 + \cdots + u^{n-1}X^{n-1}, \; q_0 = 1 - \alpha X^n,$$

and for $i > 0$,

$$p_i = p_{i-1} a \bar{p}_{i-1} a^{-1}, \qquad \bar{p}_i = a p_{i-1} a^{-1} \bar{p}_{i-1}, \qquad q_i = q_{i-1}^2.$$

Then:

(a) $p_i(0) = \bar{p}_i(0) = q_i(0) = 1$,

(b) $C_X(p_i) = -[u, a]^{(i)} a^{-i} = -C_X(\bar{p}_i)$.

Further, suppose that $N_k \lhd N_{k-1} \lhd \cdots \lhd N_0 = D^*$ is a subnormal series, that $a \in N_k$ and that $\lambda \in Z$ is such that $\lambda^n \alpha \neq 1$, and define elements $d_i = d_i(\lambda) = d_i(a, u; \lambda) = p_i(\lambda)/q_i(\lambda)$. Then:

(c) $d_i(\lambda) \in N_i$, $i = 0, 1, \ldots, k$,

(d) $d_i(\lambda)^{-1} = \bar{p}_i(\lambda)/q_i(\lambda)$.

PROOF. All the proofs are by induction on i. The case $i = 0$ is trivial for (a), (b), (c), while (d) follows from the identity

$$1 - \lambda^n \alpha = (1 - \lambda)(1 + \lambda u + \cdots + \lambda^{n-1} u^{n-1}).$$

So assume $i > 0$. Again (a) is trivial, and using (a) we compute:

$$C_X(p_i) = C_X(p_{i-1}) + a C_X(\bar{p}_{i-1}) a^{-1} = -u^{(i-1)} a^{-(i-1)} + a u^{(i-1)} a^{-(i-1)} a^{-1} = -u^{(i)} a^{-i},$$

and a similar computation yields $C_X(\bar{p}_i)$. Further, observing that $q_i(\lambda) \in Z^*$, with a liberal use of the inductive hypothesis we get:

$$d_i(\lambda) = p_i(\lambda)/q_i(\lambda) = (p_{i-1}(\lambda)/q_{i-1}(\lambda)) a (\bar{p}_{i-1}(\lambda)/q_{i-1}(\lambda)) a^{-1} = d_{i-1} a d_{i-1}^{-1} a^{-1} \in N_i$$

and

$$d_i(\lambda)^{-1} = (d_{i-1}, a)^{-1} = (a, d_{i-1}) = \bar{p}_i(\lambda)/q_i(\lambda). \qquad \square$$

The next results form a piece of folklore we have found useful to make explicit.

LEMMA 2.1. *If $a, u \in D$ are such that $[u, a] \neq 0$ and there exists a $k > 0$ such that $[u, a]^{(k)} = 0$, then there exists an $x \in D$ such that $a^{-1} x a = x + 1$.*

PROOF. Choose $r > 0$ such that $[u, a]^{(r)} \neq 0$ but $[u, a]^{(r+1)} = 0$ and set $x = a[u, a]^{(r-1)}([u, a]^{(r)})^{-1}$. Then $[x, a] = a$ and the result follows. $\square$

LEMMA 2.2. *Suppose that the polynomial $p \in D[X]$ maps infinitely many elements of Z to Z. Then $p \in Z[X]$.*

PROOF. With $p = a_0 + a_1X + \cdots + a_nX^n$, let $\lambda_0, \ldots, \lambda_n$ be distinct elements of Z and let $p(\lambda_i) = z_i$. Then

$$(a_0, a_1, \ldots, a_n) = (z_0, z_1, \ldots, z_n)((\lambda_i^j))^{-1},$$

where $((\lambda_i^j))$ is a Vandermonde matrix. □

LEMMA 2.3. *For every $a \in N - Z$, there is an N-conjugate u of a such that $(u, a) \neq 1$.*

PROOF. Deny. Let N_1 be the group generated by the N-conjugates of a; thus a is in the center of N_1. Since $N_1 \lhd N \lhd \lhd D^*$, Stuth [12, Thm. 2] implies that $a \in Z$, a contradiction. □

We can now present a simplified proof of a theorem of Herstein [5], which also illustrates the use of the Lemmas.

THEOREM 2.4. *If $N \lhd \lhd D^*$ is periodic, then $N \subseteq Z^*$.*

PROOF. We begin with a quick run through some of Herstein's arguments. The proof is by contradiction, hence we assume that N contains a noncentral element a.

Since $a^n = 1$ for some $n > 2$ (as $a^2 = 1$ implies $a = \pm 1$), we can choose $r > 1$ such that the map $a \to a^r$ induces a nontrivial automorphism of $Z(a)$. By the Noether–Skolem theorem, there is a $w \in D^*$ such that $w^{-1}aw = a^r$. Thus, for some $k > 0$ $w^kaw^k = a$, and

$$D_1 = \left\{ \sum \alpha_{ij}a^iw^j \,\middle|\, 0 \leqq i < n,\ 0 \leqq j < r,\ \alpha_{ij} \in Z(w^r) \right\}$$

is a finite dimensional $Z(w^k)$-division-algebra (see also Lemma 3.2). Now $N \cap D_1^*$ is subnormal in D_1^*, periodic, and contains the noncentral element a. Thus, by substituting D_1 for D, $N \cap D_1^*$ for N, we obtain the additional hypothesis that $[D : Z] < \infty$.

By Lemma 2.3, there exists $u \in N$ such that $(a, u) \neq 1$. Let G be the subgroup of N generated by $\{a, u\}$. As $[D : Z] < \infty$, G is isomorphic to a group of matrices; also it is finitely generated and periodic, hence by the affirmative answer to the Burnside problem, G is finite. Denote by F the prime field of Z and let k be the linear span of G over F. Then k is a finite-dimensional F-division-algebra, and Wedderburn's Theorem implies that $F = \mathbf{Q}$.

We refresh notation again by substituting k for D, $k \cap N$ for N, so that now D is a finite-dimensional $\mathbf{Q}$-division algebra.

Let $m = [D : \mathbf{Q}]$, so that D^* can be identified with a subgroup of $GL(m, \mathbf{Q})$. If $x \in D^*$ is periodic of order n, the degree $\phi(n)$ of the cyclotomic polynomial of index n must be at most m. As $\lim_{r\to\infty} \phi(r) = \infty$ there exists an $n > 0$ such that every periodic element of D^* satisfies $x^n = 1$.

We now apply the Descending Lemma to a, u. For some k, for each $\lambda \in \mathbf{Q} - \{\pm 1\}$, we have that $p_k(\lambda)/q_k(\lambda) \in N$ so that $p_k(\lambda)^n / q_k(\lambda)^n = 1$. Hence, we may apply Lemma 2.2 to conclude that $p_k^n \in Z[X]$. As $p_k(0) = 1$, we obtain

$$C_X(p_k^n) = nC_X(p_k) = -n[u, a]^{(k)}a^{-k} \in Z,$$

thus $[u, a]^{(k+1)} = 0$. By Lemma 2.1, there is an x such that $axa = x + 1$, therefore, $x = a^{-n}xa^n = x + n$, the final contradiction as $n > 0$. □

3. Special elements entailing free groups

One of the authors has proved [2] that Conjecture 2 is true when D is finite-dimensional over Z. This will be used as a basic tool, so we restate it in a convenient form.

LEMMA 3.1. *Suppose that D' is a subdivision ring of D, finite-dimensional over its center Z'. If $N \cap D'$ is not in the center of D', then N contains a free group.*

Incidentally, using this Lemma at appropriate places, one can change the proof of Theorem 2 of Lichtman [9], thus extending his result to subnormal subgroups of D^*.

Our main subdivision ring producing tool is the following, whose proof is left to the reader.

LEMMA 3.2. *Let R be a (commutative) subfield of D and suppose that $x \in D$ induces by conjugation a nontrivial automorphism of R, of finite order. Then the skew subfield $R(x)$ is finite-dimensional over its center.*

At this point, we can see several instances in which special elements in N entail a free group in N.

THEOREM 3.3. *If $a \in N$ is algebraic over Z and the field $Z(a)$ has a nontrivial Z-automorphism, then N contains a free group.*

PROOF. By the Noether–Skolem theorem, there exists an element $b \in D^*$ such that the given automorphism is induced by conjugation by b. Nontriviality of the automorphism means that $(b, a) \neq 1$, and a algebraic implies that the automorphism has finite order. By Lemma 3.2, $Z(a, b)$ is a finite-dimensional division ring, and since $a \in N \cap Z(a, b)$, Lemma 3.1 yields the result. □

COROLLARY 3.4 [2]. *If N contains a noncentral torsion element, then it also contains a free group.*

PROOF. If $a \in N - Z$ and $a^n = 1$, then $n > 2$ and for r coprime with n, $a \to a^r$ induces a nontrivial automorphism of $Z(a)$. □

A further improvement of 3.4, suggested by several results in the literature, and which is the gist of most of Section 5 is stated as a conjecture, weaker than Conjecture 2.

CONJECTURE 3. *If $N/N \cap Z^*$ contains a torsion element, then N contains a free group.*

PROPOSITION 3.5. *If Z is an absolute field, or if it contains all roots of* 1, *then Conjecture* 3 *holds for N.*

PROOF. In the first case, $a^n \in Z^*$ implies that a is torsion. In the second, if $a^n = \alpha \in Z$, $Z(a)$ is a splitting field of $X^n - \alpha$, whence the result follows from 3.3. □

PROPOSITION 3.6. *If* $\operatorname{char} D = p > 0$ *and N contains a noncentral element x such that x^p is central, then N contains a free group.*

PROOF. Choose y such that $[y, x] \neq 0$. From a well known calculation, $[y, x]^{(p)} = [y, x^p] = 0$. Lemma 2.1 yields a w such that $x^{-1}wx = w + 1$. Applying Lemma 3.2 to x and $R = Z(w)$ and then Lemma 3.1 yields the result. □

The exponent 2 yields the desired result irrespective of characteristic. Indeed, a slightly better fact is true (and compare [5, Thm. 4]):

PROPOSITION 3.7. *If N contains a noncentral element satisfying a quadratic polynomial over Z, then N contains a free group.*

PROOF. Call that element x. If x is separable over Z, clearly $Z(x)$ has a nontrivial automorphism, and the result follows from Theorem 3.3. Otherwise we are under the hypothesis of 3.6 with $p = 2$. □

As observed in the last paragraph of [5]:

COROLLARY 3.8. *D^*/Z^* contains no nontrivial subnormal p-subgroup, when* $p = \operatorname{char} D$ *or* $p = 2$.

PROOF. If there was such a group it would be of form N/Z^* with $Z^* \subsetneq N \lhd\lhd D^*$. But 3.6 or 3.7 would imply that N, hence N/Z^* would contain a free group. □

We use an argument of Lichtman [9] to prove:

PROPOSITION 3.9. *If $Z^* \subset N$ and N/Z^* contains a nonabelian finite group, then N contains a free group.*

PROOF. Let G/Z^* be that group, with $G \leqq D^*$. Then the linear span $Z[G]$ is a finite-dimensional Z-algebra, spanned by a set of coset representatives of G/Z^*. Thus it is a subdivision ring of D, which allows the application of Lemma 3.1. □

4. *SN*-words

Recall that a word w in the free group on n generators is said to be *noncentral* on a group G if the word map $G^n \to G$ defined by substitution into w is not into the center of G. And w is said to be an N-word (SN-word) if for every division ring D (any noncentral $N \lhd\lhd D^*$), w is noncentral on $D^*(N)$. The meaning of being an N-word or SN-word for some class of rings shall be understood in the obvious way.

THEOREM 4.1. *If $N \lhd\lhd D^*$ and a word w is noncentral on N, then for some $g_1, \ldots, g_n \in N$, $w(g_1, \ldots, g_n)$ has infinite order.*

PROOF. Choose a noncentral element in the image of w. There is nothing to prove unless it is torsion. In this case, Corollary 3.4 shows that N contains a free group on infinitely many generators. Substituting distinct generators into w yields the desired result. □

This Theorem leads to a variety of specializations, since we are able to produce several SN-words, or N-words.

THEOREM 4.2. *Any word that is not in the commutator subgroup of the free group is SN. Further, let, u, v be SN-words, and let w be any word. Then the following are SN-words:*

(a) *uw, provided u and w have no letters in common,*

(b) *$w^{-1}uw$, u^{-1},*

(c) *(u, v), provided u and v have no letters in common,*

(d) *u^2,*

(e) *u^p, for division rings of characteristic $p > 0$.*

PROOF. Let w be a word not in the commutator of the free group. Suppose that some noncentral $N \lhd \lhd D^*$, is mapped by w into its center. There exists a letter x such that substituting in w all other letters by 1, yields the word x^n, for some nonzero integer n. It follows that $a^n \in Z^*$ for every $a \in N$, and that contradicts Theorem 5.2, which we prove later.

Parts (a) and (b) are trivial. To prove (c), let $N \lhd \lhd D^*$ as usual. Denote by U the image of the map u on N and by V the image of v on N. The subgroups $\langle U \rangle$, $\langle V \rangle$ these sets generate are invariant in N, hence subnormal in D^*, and noncentral as u and v are SN. By Lemma 3 of [12], the group $(\langle U \rangle, \langle V \rangle)$ is noncentral. As U and V are normal sets in N, $(\langle U \rangle, \langle V \rangle)$ is generated by the commutators (a, b) with $a \in U$, $b \in V$; hence, for some $a \in U$, $b \in V$, (a, b) is noncentral, and (a, b) is in the image of (u, v).

To prove (d), let a be a noncentral image of u on N; if a^2 is noncentral we are done, else we apply u to generators of an infinitely generated free subgroup of N, which exists by Theorem 3.7. The proof of (e) is similar. □

The following is due to Amitsur [1, Thm. 19]:

THEOREM. *If Z is infinite then D^* satisfies no nontrivial relation $w = 1$; that is, the word map w on D^* is not constant.*

COROLLARY 4.3. *Any word is noncentral for division rings with infinite center.*

PROOF. If w is central for D^*, and x is a new letter, then (w, x) maps S^* to 1. □

We close the section with a couple of more contrived examples of SN-words. The first one is a small perturbation in an argument of Herstein [3].

PROPOSITION 4.4. *The word $(xy)^n x^{-n} y^{-n}$, $n \geqq 1$ is SN.*

PROOF. Suppose false, and let us consider a group $N \lhd \lhd D^*$ on which this word maps into the center. Without loss, we may assume that $Z^* \subseteq N$. Thus, we have that in N/Z^*, $(xy)^n = y^n x^n$ for every x, y. Therefore, for every x, y:

$$y^{n+1} x^{n+1} = (yx)^{n+1} = yx(yx)^n = yxx^n y^n = yx^{n+1} y^n,$$

hence $y^n x^{n+1} y^{-n} x^{-(n+1)} = 1$ in N/Z^*. This is impossible, since by Theorem 4.2, $y^n x^{n+1} y^{-n} x^{-(n+1)}$ is SN. □

PROPOSITION 4.5. *Let w be an SN-word, r, n positive integers and let x be a letter not occurring in w. Then $(w, x^r)^n$ is SN, provided* char $D = p > 0$ *and $p \nmid r$.*

PROOF. It is enough to consider the case where $p \nmid n$ since the general case will follow from (4.2.e). For a contradiction, suppose that $(w, x')^n$ maps N to its center, N being noncentral.

Let w_1 be a copy of w with entirely new letters, also distinct from x. By 4.2, $(w_1, (w, x'))$ is noncentral, hence there exist $a, u \in N$, a in the image of w (hence of w_1), u in the image of (w, x') such that $(a, u) \neq 1$. By our temporary hypothesis $u^n \in Z$. If Z is finite, Proposition 3.5 implies that N contains a free group, whence $(w, x)^n$ cannot map N into its center. Hence Z is infinite.

We apply the Descending Lemma to a, u, and obtain suitable polynomials, p_k, $\bar{p}_k$, q_k and elements $d_k(\lambda) \in N$, of which the subscript k will be dropped. Further, since a is in the image of w, $(a, d(\lambda)^r)^n \in Z$ for each possible λ. Therefore, $s = ap^r a^{-1} \bar{p}^r$ is a polynomial mapping infinitely many central elements to the center, and by Lemma 2.2, its coefficients are central. In particular,

$$C_X(s) = aC_X(p^r)a^{-1} + C_X(\bar{p}^r) = -ra[u, a]^{(k)}a^{-k}a^{-1} + r[u, a]^{(k)}a^{-k}$$
$$= r[u, a]^{(k+1)}a^{-(k+1)} \in Z,$$

and as $r \neq 0$, $[u, a]^{(k+2)} = 0$. By Lemma 2.1, there is a t such that $a^{-1}ta = t + 1$. Hence conjugation by a induces a nontrivial automorphism of $Z(t)$, of finite order p. Combining Lemmas 3.2 and 3.1 with the fact that $a \in N - Z^*$ yields that N contains a free group, and the original word has to be noncentral. □

A slight modification at this proof's end would make it work in characteristic 0 provided D was algebraic over Z. In general for characteristic 0 we only have the following result, whose proof, on the lines of Theorem 5.2, is left to the reader.

PROPOSITION 4.6. *If m, n, r are nonzero integers, then $(x^m, y^n)^r$ is SN for division rings of characteristic* 0.

5. Commutativity theorems

We begin with a restatement of Theorem 4.1.

PROPOSITION 5.1. *If w is an SN-word and for every substitution $w(x)$ in $N \lhd \lhd D^*$ there is a $n > 1$ such that $w(x)^n = 1$, then $N \leqq Z^*$.* □

If Conjecture 3 is true, the stronger result will hold that if a *SN* word maps a subnormal subgroup of D^*/Z^* to torsion elements, then that subgroup is trivial. Special cases of Conjecture 3 yield special cases of the extension: it holds when Z is absolute or when the period of each $w(x) \bmod Z^*$ has as prime divisors only 2 and char Z. Another special case has been announced [11], that the extension holds for D^* provided Z is uncountable.

A special case of the proposed extension is that periodic subnormal subgroup of D^*/Z^* must be trivial; this was proved by Herstein [6] in case Z is uncountable. Without cardinality restrictions, we show it for groups of bounded period.

THEOREM 5.2. *Suppose that $N \lhd \lhd D^*$ and there exists an integer $n > 0$ such that $x^n \in Z$ for every $x \in N$. Then $N \subseteq Z^*$.*

PROOF. By 3.5, it is enough to prove this when Z is infinite (and non-absolute). Further, we choose n minimal such that the hypothesis is satisfied, and then, if $\operatorname{char} D = p > 0$, Proposition 3.6 implies that $p \nmid n$.

Suppose that N is noncentral, for a contradiction. Then, there exist $a, u \in N$ such that $(a, u) \neq 1$. Applying the Descending Lemma, we obtain polynomials $p = p_k(a, u, X) \in D[X]$, $q \in Z[X^n]$ such that for infinitely many $\lambda \in Z$, $p(\lambda)/q(\lambda) \in N$. Then, for infinitely many $\lambda \in Z$, $p^n(\lambda) \in Z$, whence by Lemma 2.2, $C_X p^n = nC_X(p) \in Z$, and as $n \neq 0$, $C_X(p) = -[u, a]^{(k)}a^{-k} \in Z$. It follows that $[u, a]^{(k+1)} = 0$, hence by Lemma 2.1, there is a w such that $a^{-1}wa = w + 1$. As $a^n \in Z$, $w = a^{-n}wa^n = w + n$, whence $n = 0$ in D, a contradiction. □

This result self-extends, by an argument due to Kaplansky.

COROLLARY 5.3. *Suppose that there exists a nonconstant polynomial $f \in Z[X]$ such that $f(x) \in Z$ for every $x \in NZ^*$. Then $N \subseteq Z^*$.*

PROOF. If Z is absolute, the hypothesis implies that N is periodic, whence the result follows from Theorem 2.4. Hence we may suppose that there is a central element u of infinite order. Choose a polynomial f satisfying the hypothesis, of minimum degree. Now, $g(X) = f(X) - u^n f(u^{-1}X)$, where $n = \deg f$, is also central on N, and has smaller degree, whence $g = 0$. It follows that $f(X) = X^n + a$, and the preceding theorem yields the result. □

The same use of the Descending Lemma, coupled with the pigeonhole principle will yield another proof of the aforementioned theorem of Herstein. Indeed, it is easy to extend [6, Thm. 1]:

PROPOSITION 5.4. *If Z is uncountable and the word (x, y) maps N/Z^* to torsion elements, where $Z^* \subseteq N \lhd \lhd D^*$, then $N = Z^*$.*

PROOF. Suppose that $N \not\subseteq Z^*$. Again, by 3.6, the order of a noncentral element of $N \bmod Z^*$ cannot be divisible by $\operatorname{char} D$. Choose a noncentral commutator a, and an N-conjugate u such that $(a, u) \neq 1$. By hypothesis, there is an $m > 1$, such that $a^m = u^m \in Z$. Using the Descending Lemma we find torsion mod Z^* elements $d_{k+1}(\lambda) = (d_k(\lambda), a)$, with $d_k(\lambda) \in N$, for uncountably many λ. Thus, some $n \geqq 1$ is the order of $d_{k+1}(\lambda)$ mod Z^* for infinitely many λ. As in the proof of 5.2 we get a contradiction. □

A further approximation to either of the conjectures would be a possible generalization of Amitsur's Theorem 4.3 for noncentral subnormal subgroups of D^*. We did not advance much in this direction. The next results could be contrived to statements that certain words are noncentral for division rings satisfying very restrictive conditions. To avoid misunderstandings, we consider the trivial group to be torsion free.

PROPOSITION 5.5. *Suppose that* $\operatorname{char} D = 0$. *Let* n, m *be relatively prime integers, and leter,* s *be integers such that* $0 \neq r.n \neq s.m$. *Then, if the relation* $(x^n, y^n)^r = (x^m, y^m)^s$ *holds in* N/Z^*, N/Z^* *is torsion free.*

PROOF. Suppose false, that is, the relation holds and there is an $a \in N - Z$ such that $a^p \in Z$, for some $p > 1$. By Proposition 4.6, $s.m \neq 0$.

Choose u as in Lemma 2.3 and apply the Descending Lemma, thus obtaining polynomials $p_k(a, u; X)$, and so on. The given relation can be written as: $(x^n, y^n)^r.(y^m, x^m)^s \in Z$ for every $x, y \in N$. Substituting $x = a$, $y = d_k(\lambda)$ and reasoning as in earlier proofs, we deduce that:

$$r(-na^n[u,a]^{(k)}a^{-k}a^{-n} + n[u,a]^{(k)}a^{-k})$$
$$+ s(-m[u,a]^{(k)}a^{-k} + ma^m[u,a]^{(k)}a^{-k}a^{-m}) \in Z.$$

Actually one must split the argument in separate cases, depending on the signs of the numerical parameters, in order to decide where to use p_k or $\bar{p}_k$; however, the formula above is the common result for all cases.

A new derivation yields, with $w = [u, a]^{(k+1)}$:

$$sma^m wa^{-m} - rna^n wa^{-n} - (sm - rn)w = 0.$$

To finish the proof we show that $[u, a]^{(k+2)} = [w, a] = 0$; this together with Lemma 2.1 yields the desired contradiction.

Now, if $[w, a] \neq 0$, let V be the $\mathbf{Q}$-submodule of D spanned by $\{w, awa^{-1}, \ldots, a^{p-1}wa^{-(p-1)}\}$, and let T be the automorphism of V given by $T(x) = axa^{-1}$. Clearly $T^p = I$, while the last equation implies that $smT^m - rnT^n = (sm - rn)I$. Choose a (complex) eigenvalue θ of T. Then $sm\theta^m - rn\theta^n = sm - rn$, that is $\theta^n(\theta^{m-n} - rn/sm) = 1 - rn/sm$. As $|\theta| = 1$, this implies $|\theta^{m-n} - rn/sm| = 1 - rn/sm$, and as $rn \neq sm$, we conclude that $\theta^{m-n} = 1$ (just draw a picture). It follows now that $\theta^n = 1 = \theta^m$. As n, m are coprime, $\theta = 1$.

That means that $T = 1$, whence $awa^{-1} = w$, contradicting the supposition that $[w, a] \neq 0$. □

The same method, aided by a careful bookkeeping, yields:

PROPOSITION 5.6. *Let sequences* $n_1, \ldots, n_l$, $m_1, \ldots, m_l$, $r_1, \ldots, r_l$ *of nonzero integers be given and suppose that the only complex zero of the rational function* $\Sigma_i r_i m_i (X^{n_i} - 1)$ *which is also a root of* 1 *is* $X = 1$. *Then if* $\mathrm{char}\, D = 0$ *and the word* $\Pi_i (x^{n_i}, y^{m_i})^{r_i}$ *maps* N *to its center, then* N/Z^* *is torsion free.*

REFERENCES

1. S. A. Amitsur, *Rational identities and applications to algebra and geometry*, J. Algebra **3** (1966), 304–359.
2. J. Z. Gonçalves, *Free groups in subnormal subgroups and the residual nilpotence of the group of units of group rings*, Can. Math. Bull. **27** (1984), 365–370.
3. I. N. Herstein, *Power maps in rings*, Mich. Math. J. **8** (1961), 29–32.
4. I. N. Herstein, *Noncommutative rings*, The Carus Math. Monographs n. 15, The Math. Assoc. of America, 1968.
5. I. N. Herstein, *Multiplicative commutators in division rings*, Israel J. Math. **31** (1978), 180–188.
6. I. N. Herstein, *Multiplicative commutators in division rings II*, Rend. Circ. Math. di Palermo, Serie II, Tomo **XXIX** (1980), 485–489.
7. I. N. Herstein and W. R. Scott, *Subnormal subgroups of division rings*, Can. J. Math. **15** (1963), 80–83.
8. A. I. Lichtman, *On subgroups of the multiplicative group of skew fields*, Proc. Amer. Math. Soc. **63** (1977), 15–16.
9. A. I. Lichtman, *Free subgroups of normal subgroups of the multiplicative group of skew fields*, Proc. Amer. Math. Soc. **71** (1978), 174–178.
10. A. I. Lichtman, *On normal subgroups of the multiplicative group of skew fields generated by a polycyclic-by-finite group*, J. Algebra **78** (1982), 548–577.
11. L. Makar-Limanov and P. Malcolmson, *Words periodic over the center of a division ring*, Abstract 810-16-11, Abs. Amer. Math. Soc. **2** (1984), 160.
12. C. Stuth, *A generalization of the Brauer–Cartan–Hua theorem*, Proc. Amer. Math. Soc. **15** (1964), 211–217.
13. J. Tits, *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270.